Final Exam
Inverse problems and high dimension
Convexity & Sparsity

March 27th 2023
2pm to 4pm

This subject has two exercises. Non-master students will be evaluated on all the exercise 1 and the four first questions of exercise 2. The exercises are independent. Master students will be evaluated on all the subject. Questions with a coffee cup ☕ are bonus questions and would be granted extra-points in case of good answers.

This subject has 3 pages and you have two hours. **You are kindly asked to write your answers on two separate sheets, one for exercise 1 and one for exercise 2** in french or in english.

Exercise 1

The goal of this exercise is to investigate recovery guarantees for OMP based on the restricted isometry property. In the following, $\mathbf{A}$ is a matrix of size $m \times n$ where $m < n$, $[\![n]\!]$ denotes the set of all integers between 1 and n, $S \subseteq [\![n]\!]$ is a set of indices, $s = |S|$, and T is its complement in $[\![n]\!]$. The pseudo-inverse is denoted $(\cdot)^\dagger$. We remind that the following are equivalent:

- every vector $\mathbf{x}$ with support included in the set S (i.e., $\texttt{supp}(\mathbf{x}) \subseteq S$) is recovered by OMP with s steps performed on $\mathbf{y} := \mathbf{A}\mathbf{x}$.
- $\mathbf{A}$ satisfies the *exact recovery property* (ERC) with respect to the set S,

$$\max_{j \notin S} \|(\mathbf{A}_S)^\dagger \mathbf{a}_j\|_1 < 1. \tag{1}$$

We also recall that the restricted isometry constant of order k of matrix $\mathbf{A}$, denoted $\delta_k = \delta_k(\mathbf{A})$, is the smallest real number $\delta > 0$ such that

$$(1-\delta)\|x\|_2^2 \leq \|\mathbf{A}x\|_2^2 \leq (1+\delta)\|x\|_2^2, \quad \forall x \in \mathbb{R}^n \text{ such that } \|x\|_0 \leq k.$$

1. Assume that $\delta_s(\mathbf{A}) < 1$.

   (a) Show that for any vector v such that $\texttt{supp}(v) \subseteq S$ we have

   $$(1-\delta_s)\|v\|_2 \leq \|\mathbf{A}_S^\top \mathbf{A}_S v_S\|_2 \leq (1+\delta_s)\|v\|_2.$$

   (b) Deduce that for any vector u we have

   $$\frac{1}{1+\delta_s}\|u_S\|_2 \leq \|(\mathbf{A}_S^\top \mathbf{A}_S)^{-1} u_S\|_2 \leq \frac{1}{1-\delta_s}\|u_S\|_2$$

2. Consider two sets such that $S_1 \cap S_2 = \emptyset$ and $\delta_{s_1+s_2}(\mathbf{A}) < 1$ where $s_i := |S_i|$.

   (a) Show that if $\texttt{supp}(u) \subseteq S_1$, $\texttt{supp}(v) \subseteq S_2$ then

   $$|\langle \mathbf{A}_{S_1} u_{S_1}, \mathbf{A}_{S_2} v_{S_2} \rangle| \leq \delta_{s_1+s_2}\|u\|_2\|v\|_2.$$

(b) Show that if $\mathtt{supp}(v) \subseteq S_2$, then

$$\|\mathbf{A}_{S_1}^\top \mathbf{A}_{S_2} v_{S_2}\|_2 \le \delta_{s_1+s_2}\|v\|_2.$$

3. Assume that $\delta_{s+1} < 1$

(a) Consider $j \in T$. Show that

$$\|(\mathbf{A}_S)^\dagger a_j\|_2 \le \frac{\delta_{s+1}}{1-\delta_s}$$

(b) Explain why this implies that

$$\|(\mathbf{A}_S)^\dagger a_j\|_1 \le \frac{\sqrt{s}\delta_{s+1}}{1-\delta_{s+1}}$$

4. Express a sufficient condition, based on δ_{s+1}, to ensure exact recovery of s-sparse vectors with s steps of OMP. The condition should read

$$\delta_{s+1}(\mathbf{A}) < c_s$$

with an appropriate choice of c_s.

5. (☕ Bonus) Can the condition be tightened ?

Exercise 2

In the following, $\mathbf{A}$ is a matrix of size $m \times n$ where $m < n$, $[\![n]\!]$ denotes the set of all integers between 1 and n. For $\mathbf{x} \in \mathbb{R}^n$, let $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}$ for some $\mathbf{e}$ satisfying $\|\mathbf{A}^\top \mathbf{e}\|_\infty \le \eta$ for some $\eta > 0$, where $\mathbf{A}^\top$ denotes the transpose matrix of $\mathbf{A}$. Let $\mathbf{x}^\star$ be a minimizer of the *Dantzig selector* defined as

$$\min_{\mathbf{z}\in\mathbb{R}^n} \|\mathbf{z}\|_1 \text{ subject to } \|\mathbf{A}^\top(\mathbf{y} - \mathbf{A}\mathbf{z})\|_\infty \le \eta\,. \tag{2}$$

1. Show that a minimizer $\mathbf{x}^\star$ of Program (2) always exists.

The goal of this exercise is to prove that the Dantzig selector (2) satisfies robustness and stability under the ℓ_1-robust null space property (RNSP), namely it holds

$$\|\mathbf{x} - \mathbf{x}^\star\|_1 \le C\sigma_s(\mathbf{x})_1 + D\, s\,\eta\,, \tag{3}$$

where $C, D > 0$ are constants depending on the RNSP constants ρ and τ (see below), $s \ge 1$, and

$$\sigma_s(\mathbf{x})_1 = \min_{\mathbf{z}\in\mathbb{R}^n} \|\mathbf{x} - \mathbf{z}\|_1 \text{ subject to } \|\mathbf{z}\|_0 \le s\,,$$

is the best s-term approximation error of $\mathbf{x}$ for the ℓ_1-norm.

2. Prove that there exists $S \subseteq [\![n]\!]$, $s = |S|$, such that $\sigma_s(\mathbf{x})_1 = \|\mathbf{x}_T\|_1$, where T is the complement of S in $[\![n]\!]$. Is S unique?

We recall that the ℓ_1-robust null space property (RNSP) of order s with constant $0 < \rho < 1$ and $\tau > 0$ is defined as

$$\|\mathbf{v}_S\|_1 \le \rho\|\mathbf{v}_T\|_1 + \tau\sqrt{s}\|\mathbf{A}\mathbf{v}\|_2 \tag{4}$$

for all $\mathbf{v} \in \mathbb{R}^n$ and all $S \subseteq [\![n]\!]$ set of indices, $s = |S|$, and T is its complement in $[\![n]\!]$. Consider $\mathbf{h} := \mathbf{x}^\star - \mathbf{x}$.

3. Prove that

$$\|\mathbf{A}\mathbf{h}\|_2^2 \le \|\mathbf{A}^\top(\mathbf{A}\mathbf{h} - \mathbf{y} + \mathbf{y})\|_\infty \|\mathbf{h}\|_1\,,$$

and deduce that

$$\|\mathbf{A}\mathbf{h}\|_2^2 \le 2\eta\|\mathbf{h}\|_1\,. \tag{5}$$

4. Show that $\|x^\star\|_1 \le \|x\|_1$, deduce that

$$\|\mathbf{x}_T^\star\|_1 \le \|\mathbf{x}_T\|_1 + \|\mathbf{h}_S\|_1\,,$$

and that

$$\|\mathbf{h}_T\|_1 \le 2\|\mathbf{x}_T\|_1 + \|\mathbf{h}_S\|_1\,. \tag{6}$$

MASTER students only

5. From (5) and (6) prove that for all $t > 2\eta$,

$$\|\mathbf{A}\mathbf{h}\|_2^2 + (t-2\eta)\|\mathbf{h}_T\|_1 \leq (t+2\eta)\|\mathbf{h}_S\|_1 + 2t\|\mathbf{x}_T\|_1 \,. \tag{7}$$

   and deduce that

$$\gamma_t\|\mathbf{h}\|_1 \leq -\frac{1}{t+2\eta}\|\mathbf{A}\mathbf{h}\|_2^2 + (1+\gamma_t)\|\mathbf{h}_S\|_1 + \frac{2t}{t+2\eta}\|\mathbf{x}_T\|_1 \,, \tag{8}$$

   where $\gamma_t := \frac{t-2\eta}{t+2\eta}$.

6. For $0 < \rho < 1$ defined in (4), prove that there exists $1/2 < \alpha < 1$ such that

$$\rho = \frac{1-\alpha}{\alpha} \,.$$

7. Prove that there exists $t > 0$ such that $\alpha\gamma_t - (1-\alpha) > 0$.

We consider that t satisfies this latter condition in the following questions.

8. Using (4) and (8), deduce that

$$(\alpha\gamma_t - (1-\alpha))\|\mathbf{h}\|_1 \leq -\frac{1}{t+2\eta}X^2 + \alpha(1+\gamma_t)\tau\sqrt{s}X + \frac{2t}{t+2\eta}\|\mathbf{x}_T\|_1 \,. \tag{9}$$

   where $X := \|\mathbf{A}\mathbf{h}\|_2$.

9. Using (9), prove (3).